ITP Budapest Report No. 523
hep-th//9702183

On the free field realization of WBC_n algebras

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February 26, 1997

Abstract

Defining the WBC_n algebras as the commutant of certain screening charges a special form for the classical generators is obtained which does not change under quantisation. This enables us to give explicitly the first few generators in a compact form for arbitrary WBC_n algebras.

1 Introduction

The first definition of the W-algebras was given by Fateev and Lukyanov in [1]. They defined the classical W-algebras using the Miura transformation, consequently the algebras were presented in their free field form. In order to find the quantum analogue they quantised the free fields and normal ordered the classical expressions. Then the screening charges, operators that commute with the W-generators, were identified since both the screening operators and the W-generators are necessary in the investigation of the representation theory. Unfortunately this quantisation procedure works only for the A_n and D_n series, but fails to work for C_n and B_n. To avoid this problem they modified the screening operators and the W-generators simultaneously for the B_n algebras but in doing so the correspondence to the classical theory is lost.

Using an alternative definition of the quantum W-algebras Feigin and Frenkel realized [2], that most naturally one can define the W-generators and so the algebras as the commutant of the screening charges. They also computed the classical counterpart and showed the relation between them: they first proved the existence of the classical generators then proved their survival at the quantum level.

Although the existence of the quantum generators is established their explicit form is still unknown in the general case. In fact, Lukyanov and Fateev's quantum expressions for A_n and D_n still work, since in this case the polynomial form of the W-generators does not change only the coefficients get quantum corrections. The aim of this paper is to find similar expressions for the missing B_n and C_n cases, ie. to find such a form for the generators at the classical level, which in this sense survive quantisation. Since there are applications where the W-generators are explicitly needed [10, 13] we also give explicit expressions, in a very compact form, for the first few generators of arbitrary WBC_n algebras. (See also [11] for related results).

The paper is organised as follows: in Section 2 we review the earlier results concerning the definition and quantisation of W-algebras. Then in Section 3 we consider the classical theory and find a form for the generators which turns out to be very useful at the quantum level. In Section 4 we prove that the classical form does not change and give explicit results. Afterwards we conclude in Section 5.

2 *WG* algebras as commutant of screening operators

Usually the *WG* algebras are defined as the reduction of the Kac-Moody (KM) algebras corresponding to the principal sl_2 embeddings [8]. On can show that this is equivalent to look only for the Cartan currents of the KM algebra in question and find the commutant of certain screening charges [3].

In more detail one defines the *WG* algebra as a vertex operator sub-algebra of the KM algebra's Cartan sub-algebra. The commutation relations of the modes of the ghost-modified Cartan currents [4, 9] are en-coded in

$$\hat{J}^i_1|\hat{J}^j_{-1}\rangle = \alpha_{ij}(k+h)|0\rangle = \alpha_{ij}\kappa|0\rangle, \tag{1}$$

where k denotes the level and h the dual Coxeter number. The $\hat{J}^i$ correspond to the simple roots,α_i, and $\alpha_{ij} = \alpha_i \cdot \alpha_j$ is the inner product matrix of G. The space of states is spanned by

the negative modes of the currents and the WG algebra is nothing but the only non-vanishing cohomology of the BRST charge $\sum_i Q_i$, consisting of the sum of the screening charges, Q_i,

$$Q_i = \sum_{n\leq 0} S^i_{n+1}(\hat{J}^i_-)S^i_n(\hat{J}^i_+) \quad ; \quad \sum_{n\leq 0} S^i_n(x^i_\pm)z^n = \prod_{m<0} e^{\pm\frac{x^i_{\mp m}}{m}z^m}. \tag{2}$$

The kernel of Q_i coincides with the kernel of $\oint \frac{dz}{2\pi i} : e^{\int^z \hat{J}^i(w)dw}:$.

The classical limit can be associated to $\kappa \to 0$ [2]. In this case the classical screening charges become:

$$\tilde{Q}_i = -\kappa^{-1}\lim_{\kappa\to 0} Q_i = \sum_{n<0;j} S^i_{n+1}(\hat{J}^i_-)\alpha_{ij}\frac{\partial}{\partial \hat{J}^j_n} = \sum_{n<0} S^i_{n+1}(\sum_j \alpha_{ij}h^j_-)\partial^i_n \qquad ; \quad \partial^i_{-n} = \frac{\partial}{\partial h^i_{-n}}, \tag{3}$$

where we have introduced the $\sum_i \alpha_{ij}h^j_n = \hat{J}^i_n$ variables, which correspond to the fundamental co-weights. Here, analogously to the quantum case, h^i_{-m-1} represents the $(m!)^{-1}\partial^m h^i(z)$ classical field. The classical analogue of L_{-1} is $\partial = \sum_{m<0}\sum_{i=1}^{dimg} mh^i_{-m-1}\partial^i_m$, which is the operator of the differentiation on the differential polynomials of the classical fields, $h^i(z)$.

Using the fact that $\tilde{Q}_i$-s satisfy the Serre relation of the algebra Feigin and Frenkel proved that the kernel of these operators is spanned by elements W_{e_i+1} of degree e_i+1 together with products of their derivatives. Here the e_i-s denote the exponents of the Lie algebra. Modifying the proof slightly one can show that taking a subset $I \subset \{1,\dots,n\}$, the common kernel of the $\tilde{Q}_i$ operators for $i \in I$ consists of the W-generators of the associated sub-algebra, (whose simple roots are α_i , $i \in I$), and the remaining h^j_n , $j \notin I$ variables. As a consequence the kernel of Q^i is generated by $L^{(i)}$ and h^j_n, $j \neq i$, where $L^{(i)}$ is defined as

$$\partial^2 + (h_{i+1} - h_{i-1})\partial + L^{(i)} = (\partial + h_{i+1} - h_i)(\partial + h_i - h_{i-1}), \tag{4}$$

(if $\alpha_{ij} = 2\delta_{ij} - \delta_{i-1j} - \delta_{i+1j}$). Here and from now on we abbreviate h^i_{-1} by h_i. The translation covariance of this expression motivates one to find the A_n generators in the following form:

$$(\partial - h_n)\dots(\partial + h_{i+1} - h_i)(\partial + h_i - h_{i-1})\dots(\partial + h_1) = \sum_{i=0}^n W^{A_n}_i \partial^{n-i}. \tag{5}$$

Unfortunately this translation covariance is absent in the the C_n and B_n case, for this reason we choose the following strategy. As a starting point we consider the first $n-1$ screening operators. Since they are exactly the same as in the A_n case one knows their kernel and the corresponding generators. In order to find the common kernel we need $L^{(n)} = h_nh_n - 2h_nh_{n-1} - \partial h_n$ so we combine (5) with its adjoint and obtain the correct answer:

$$(\partial - h_1)\dots(\partial + h_{n-1} - 2h_n)(\partial + 2h_n - h_{n-1})\dots(\partial + h_1) = \sum_i W^{C_n}_{2i}\partial^{2(n-i)} + \text{odd terms}. \tag{6}$$

One has a similar solution for B_n:

$$(\partial - h_1)\dots(\partial + h_{n-1} - h_n)\partial(\partial + h_n - h_{n-1})\dots(\partial + h_1) = \sum_i W^{B_n}_{2i}\partial^{2(n-i)+1} + \text{even terms}. \tag{7}$$

The quantum case is much more involved. Since the

$$Q_i = \sum_{n\leq 0} S^i_{n+1}(\hat{J}^i_-)S^i_n(\hat{J}^i_+) = \sum_{n\leq 0} S^i_{n+1}(\sum_j \alpha_{ij}h^i_-)S_n(h^i_m \to -m\kappa\partial^i_{-m}) = -\kappa\tilde{Q}_i + \kappa^2(\ldots) \quad (8)$$

quantum operators satisfy the q-Serre relations of the underlying q-deformed Lie algebra the classical W-generators survive [2]. However this means that the commutant of the subset of the Q_i operators, in harmony with the classical theory, consists of the W-generators of the associated sub-algebra and the remaining h^j , $j \neq i$ variables. To find the explicit quantum expressions one tries the classical form modified according to $L^{(i)}_q = L^{(i)}_{cl}(\partial \to (1-\kappa)\partial)$. It works for A_n:

$$(\partial_\kappa - h_n)\ldots(\partial_\kappa + h_{i+1} - h_i)(\partial_\kappa + h_i - h_{i-1})\ldots(\partial_\kappa + h_1) = \sum_{i=0}^{n} W^{A_n}_i \partial^{n-i}_\kappa \qquad ; \quad \partial_\kappa = (1-\kappa)\partial. \quad (9)$$

In the other cases this naive quantisation does not work since one has to deal with the corrections coming form the normal ordering of more then two h^i-s.

Taking into account the special form of the Q_i operators (the κ^n terms contain n differentiations) one can show that each quantum generator has the following form [5]:

$$P^\kappa = P_0 + \kappa P_1 + \ldots + \kappa^i P_i + \ldots + \kappa^{k-1}P_{k-1}, \quad (10)$$

where P_0 is the classical expression, eq. (5,6,7), and P_i contains the product of at most $k-i$ terms. Moreover the sum of the longest terms, terms without derivatives, gives a Weyl invariant polynomial.

3 Classical considerations

We have seen in the previous section that the zeroth order term in the quantum expression is the classical generator. This motivates us to start at the classical level where we set up the notations strongly suggested by the quantum problems.

Since we know how the A_n generators get quantum corrections, (9), we rewrite the classical generator (6) into to following form:

$$\sum_i W^{C_n}_{2i}\partial^{2(n-i)} + \text{odd terms} = \left(\partial^n - \partial^{n-1}W_1 + \ldots + (-1)^n W_n\right)\left(W_n + \ldots + W_1\partial^{n-1} + \partial^n\right). \quad (11)$$

In order for $(\partial + 2h_n - h_{n-1})$ to be the same as in the A_n case: $(\partial + h_n - h_{n-1})$, we rescaled h_n by a factor two. Here the W_k-s span the kernel of the $Q_1, \ldots, Q_{n-1}$ operators. Clearly $W_1 = h_n$. Moreover if one takes the $h_1 \to 0$ limit then $W_n \to 0$ and for all the others we recover the generators related to the A_{n-1} algebra.

Unfortunately in the definition above the $W^{C_n}_{2i}$ generators explicitly depend on n. We would like to redefine them n-independently. We also need a form for the $W^{C_n}_{2k}$ generator such that $W^{C_n}_{2k} \to 0$ when $W_l \to 0$, $l = n, n-1, \ldots k$. This can be achieved by the following way: define the generators by

$$(\partial^n - \partial^{n-1}W_1 + \ldots + (-1)^n W_n)(W_n + \ldots + W_1\partial^{n-1} + \partial^n) =$$
$$\partial^n\partial^n + \partial^{n-1}W^{C_n}_2\partial^{n-1} + \partial^{n-2}W^{C_n}_4\partial^{n-2} + \ldots + \partial^{n-k}W^{C_n}_{2k}\partial^{n-k} + \ldots + W^{C_n}_{2n}. \quad (12)$$

It is not completely clear at first that this definition is correct. Writing this expression into the original form (6), one can see that at each even level (12) really defines the generator, however at odd levels it is a nontrivial statement. In order to prove it first we note that

$$W_{2n}^{C_n} = \left(\partial^n - \partial^{n-1}W_1 + \ldots + (-1)^n W_n\right) W_n. \tag{13}$$

This shows that if we take the $h_1 \to 0$ limit then $W_{2n}^{C_n} \to 0$ while for all the other generators we have $W_{2k}^{C_n} \to W_{2k}^{C_{n-1}}$. This indicates that the $W_{2k}^{C_n}$ generators share the property needed above, ie. $W_{2k}^{C_n} \to 0$ when $W_l \to 0$, $l = n, \ldots, k$. It is also clear that the generators are n-independent. Now the terms at odd level have to be in the kernel of all the screening charges. This means that they have to be a linear combination of the derivatives of the $W_{2k}^{C_n}$ generators defined at even levels. However one can determine the coefficients of this combination from the $W_k W_k$ term of $W_{2k}^{C_n}$, which are exactly what is needed.

Summarizing the classical generator has the following form:

$$W_{2k}^{C_n} = \sum_{i,j,l,m} c_{ij}^{lm} \partial^i W_l \partial^j W_m, \tag{14}$$

where $i + j + l + m = 2k$ and $l \leq k \leq m$.

Here we list the first few generators since we will need their explicit form at the quantum level.

$$\begin{aligned}
W_2^{C_n} &= 2W_2 - (W_1)^2 + \partial W_1 \\
W_4^{C_n} &= 2W_4 + 3\partial W_3 - 2W_1W_3 + W_2W_2 - (\partial W_1)W_2 - W_1\partial W_2 + \partial^2 W_2 \\
W_6^{C_n} &= 2W_6 - 2W_1W_5 + 2W_2W_4 - W_3W_3 + 5\partial W_5 - 3\partial(W_1W_4) \\
&\quad + \partial(W_2W_3) + 4\partial^2 W_4 - \partial^2(W_1W_3) + \partial^3 W_3.
\end{aligned} \tag{15}$$

Similar considerations can be done also for the B_n algebras.

4 Quantum considerations

First we will show that the quantum generator has the same polynomial form as the classical one (14) only the c_{ij}^{lm} coefficients get quantum corrections:

$$W_{2k}^{C_n} = \sum_{i,j,l,m} c_{ij}^{lm}(\kappa) \partial^i W_l \partial^j W_m, \tag{16}$$

where now W_l denotes the explicitly known quantised A_n-type expression, (9,12) and normal ordering has to be understood.

The idea of the proof is to show that terms containing more then two h_n-s can be removed systematically since they are absent at the classical level, (6). As a starting point we write down the most general expression which commute with the first $(n-1)$ screening operators:

$$W_{2k}^{C_n} = \sum_{j_1,\ldots,j_l;i_1,\ldots,i_l} c_{i_1,\ldots,i_l}^{j_1,\ldots,j_l}(\kappa) \partial^{i_1} W_{j_1} \ldots \partial^{i_l} W_{j_l}. \tag{17}$$

Now we demand for it to be in the kernel of Q_n ie. to contain $L^{(n)} = h_n h_n - 2h_n h_{n-1} - (1 - 2\kappa)\partial h_n$:

$$W_{2k}^{C_n} = \sum_{i_1,\dots i_l} b^{i_1,\dots i_l}(\kappa)\partial^{i_1} L^{(n)} \dots \partial^{i_l} L^{(n)}\{\text{terms without } h_n\}. \tag{18}$$

We have to redefine this W-generator such a way that the new one contains the products of at most two W_l-s. We will do it inductively: we define a partial ordering on the space of states. We say that $\partial^{i_1} h_n \dots \partial^{i_k} h_n \partial^{i_{k+1}} h_{l_1} \dots \partial^{i_{k+s}} h_{l_s}$ is bigger than $\partial^{j_1} h_n \dots \partial^{j_m} h_n \partial^{j_{m+1}} h_{p_1} \dots \partial^{j_{m+t}} h_{p_t}$ if $k > m$ or if $k = m$ then if $s > t$. This induces an ordering among the c coefficients if we associate the $h_n h_{n-1} \dots \partial^i h_{n-j+1}$ highest grade term to $\partial^i W_j$. (We note that the restriction of this map to the highest grade is injective, moreover normal ordering and other quantum corrections contribute at lower grades only). Now consider the highest grade terms in the expressions, (17, 18), together with their W_l and $L^{(n)}$ generators. We show in the Appendix that there exist a classical W-generator with these highest grade terms. This means however that the unneeded highest grade terms can be removed by redefining the W-generator, (17). Doing this procedure from grade to grade we end up with the original form (16).

In order to compute the coefficients explicitly we will use duality [7]. It states that if one replaces h_i by $\beta\tilde{h}_i$, where $\beta^2 = \kappa$ and defines $\tilde{W}_{2k}^{C_n}(\beta) = \beta^{-2k} W_{2k}^{C_n}(\beta\tilde{h}_i)$, (and similarly for B_n) then

$$\tilde{W}_{2k}^{C_n}(\beta) = \tilde{W}_{2k}^{B_n}(-\beta^{-1}). \tag{19}$$

Since the A_n algebra is self-dual this transformation acts only on the c coefficients in the expression (16). In more detail this means that rescaling the generators into their original forms the coefficient of the κ^{i+j-m} term in $c_{ij}^{kl}(\kappa)$ for the C_n algebra becomes the coefficient of the $(-\kappa)^m$ term in the analoguous expression for B_n.

The duality transformation makes it possible to give the quantum generator at level two:

$$W_2^{C_n} = 2W_2 - (W_1)^2 + (1 - 2\kappa)\partial W_1. \tag{20}$$

This form of the generator is n-independent and thus it is the same for all the WBC_n algebras.

Lets consider the next case, at level four:

$$W_4^{C_n} = aW_4 + b\partial W_3 + cW_1 W_3 + W_2 W_2 + d\partial W_1 W_2 + eW_1 \partial W_2 + f\partial^2 W_2, \tag{21}$$

where the terms without quantum corrections are: $a = 2$ and $c = -2$. Duality restricts the others as

$$b = 3 - 4\kappa \quad ; \quad d = -1 \quad ; \quad e = -(1 - 2\kappa) \quad ; \quad f = 1 + f_1\kappa + 2\kappa^2, \tag{22}$$

where the f_1 term is still unknown. However this is not surprising since it may depend on the choice of the normal ordering of the $W_2 W_2$ term, (all the other normal orderings are trivial in (21)). We note that the W_k generators are exactly the same for the A_n, B_n and C_n algebras since the first $(n-1)$ screening charges coincide. This means that computing the normal ordering one can use any of these possibilities, moreover the normal ordering corresponding to the $N > n$ algebras. These choices are different since the normal ordering uses the inverse of the inner product matrix. The difference of the different orderings has to be in the kernel of the first $(n-1)$ generators so it can be expressed in terms of the W_l-s, moreover it contains at most

two h_n-s, so it has the form (16). Note that the induced terms and so the quantised generator $W_4^{C_n}$ may contain the classically removed W_1-s. However taking the $W_k \to 0$, $k = 2, 3, 4$ limit the generator $W_4^{C_1}$ obtained has to be proportional to $\partial^2 W_2^{C_1}$, so we can take such a linear combination of the generators $W_4^{C_n}$ and $\partial^2 W_2^{C_n}$ which does not contain the W_1 operator. We chose a normal ordering which is exceptional in the sense that it contains the fewest new terms:

$$: W_2 W_2 := (W_2 W_2)|_{C_n} + \frac{\kappa}{2}\partial^2 W_2 - \ldots , \tag{23}$$

where the dots mean that we removed the induced n-dependent W_1 terms and the subscript C_n means that we used the inner product matrix of the C_n algebra. Now the term f becomes $(1-\kappa)(1-2\kappa)$ and thus $f_1 = -3\kappa$.

Next consider the generator at level six. The terms which contain at most one derivatives are determined by duality:

$$\begin{aligned}&2W_6 - 2W_1W_5 + 2W_2W_4 - W_3W_3 + \partial(W_2)W_3 + (5-6\kappa)\partial W_5 \\ &-(3-4\kappa)\partial(W_1)W_4 - (3-2\kappa)W_1\partial W_4 + (1-2\kappa)W_2\partial W_3.\end{aligned} \tag{24}$$

The other terms we compute in the C_3 model. Doing explicitly the calculations we have:

$$\begin{aligned}&-(1-\kappa)(1-2\kappa)W_1\partial^2 W_3 - (2-\kappa)(1-2\kappa)\partial W_1 \partial W_3 \\ &\quad -2(1-\kappa)(\partial^2 W_1)W_3 + (1-\kappa)^2(1-2\kappa)\partial^3 W_3.\end{aligned} \tag{25}$$

If one computes the same terms in the C_n case the coefficients may change due to the normal ordering which depends on n. The best one can do is to define an n-independent normal ordering. However to do it in the general case one has to know the various defining relations of the W-algebras which is still missing. Concretely in the C_3 case there are induced lower level terms:

$$\begin{aligned}&\tfrac{\kappa}{12}(8\kappa-7)\partial^3 W_3 - \tfrac{\kappa}{2}\partial W_2 \partial W_2 - \tfrac{\kappa}{12}(10\kappa-7)W_1\partial^3 W_2 \\ &+\tfrac{\kappa}{4}(3-4\kappa)\partial W_1 \partial^2 W_2 + \tfrac{\kappa}{2}(2\kappa+1)\partial^2 W_1 \partial W_2 + \tfrac{\kappa}{2}(12\kappa-7)\partial^3 W_1 W_2.\end{aligned} \tag{26}$$

The generator at level six is the sum of (24), (25) and (26).

5 Conclusion

We analyzed the generators of the WBC_n algebra using the explicitly know generators of the WA_n algebra. Starting at the classical level we found a special n-independent form which does not change under quantisation. In the general case to make it more explicit and go beyond the terms determined by duality one has to compute the normal ordering explicitly for all the W-generators.

Aknowledgements

We would like to thank L. Palla and L. Fehér for the critical reading and the helpful comments. This work was supported by OTKA T016251 and OTKA F019477.

6 Appendix

The proof is very technical so we just sketch it. First we introduce the notations: In terms of the classical fields, $u_i = h_i - h_{i-1}$ (with $h_0 = 0$), the sum of the leading terms of the generator W_k, which is denoted by w_k, become the k-th elementary symmetric polynomial:

$$w_k = \sum_{i_1<\ldots<i_k} u_{i_1}\ldots u_{i_k} \tag{27}$$

If $\mathcal{A}_x$ denotes the algebra of the differential polynomials of the arbitrary classical field x and we abbreviate $\mathcal{A}_{w_1,\ldots,w_n}$ by $\mathcal{A}_w$ then the statement we have to prove is the following, (see (17,18)),

$$\mathcal{A}_w \cap \mathcal{A}_{u_n^2} = \mathcal{A}_{w_2^c,\ldots,w_{2n}^c} = \mathcal{A}_{w^c} \tag{28}$$

where w_{2k}^c denotes the sum of the leading terms of the classical generator W_{2k}^C, ie. $w_2^c = w_1^2 - 2w_2, \ldots, w_{2n}^c = (-1)^n w_n w_n$.

Clearly the algebra $\mathcal{A}_w$ is a subalgebra of $\mathcal{A}_{sym}$, the algebra of the symmetric differential polynomials in the fields u_i-s for which a polynomial basis is given by

$$p_{n_1,\ldots,n_k} = \sum_{i_1,\ldots,i_k;\mathrm{all}\neq} \partial^{n_1}u_{i_1}\ldots\partial^{n_k}u_{i_k},\ n_i \leq n_{i+1};\ k = 1,2,\ldots,n \tag{29}$$

see [12] for the details. Consequently

$$\mathcal{A}_w \cap \mathcal{A}_{u_n^2} = \mathcal{A}_w \cap \mathcal{A}_{sym^2}, \tag{30}$$

where $\mathcal{A}_{sym^2}$ denotes the algebra of the symmetric differential polynomials in the fields u_i^2-s, with polynomial basis $P_{n_1,\ldots,n_k} = p_{n_1,\ldots,n_k}(u_i \to u_i^2)$. Note that $P_{n_1,\ldots,n_k} = w_{2k}^c$ if $n_i = 0$ for all i.

Define the endomorphism φ on $\mathcal{A}_{sym}$ by

$$\varphi(p_{n_1,\ldots,n_k}) = \begin{cases} p_{n_1,\ldots,n_k} & \text{if } n_1 = n_2 = \ldots = n_{k-1} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

It is not hard to see that $\varphi : \mathcal{A}_w \to \mathcal{A}_{sym}/\mathrm{Ker}\ \varphi$ is an isomorphism.

Note also that $P_{n_1,\ldots,n_k} = \sum_{i_1,\ldots,i_k;\mathrm{all}\neq} \partial^{n_1}u_{i_1}^2\ldots\partial^{n_k}u_{i_k}^2$ is a linear combination of terms of the form $p_{i_1,\ldots,i_k}p_{i_{k+1},\ldots,i_{2k}}$, where $i_1,\ldots,i_{2k}$ is any of the permutations of the non-negative integers $j_1,\ldots,j_k,n_1-j_1,\ldots,n_k-j_k$. This shows that $P_{n_1,\ldots,n_k} \in \mathrm{Ker}\ \varphi$ except if $n_1 = n_2 = \ldots = n_{k-1} = 0$. Since $P_{0,0,\ldots,0,n_k} = \partial^{n_k}P_{0,0,\ldots,0}$ on $\mathcal{A}_{sym}/\mathrm{Ker}\varphi$, which is isomorphic to $\mathcal{A}_w$, and $P_{0,0,\ldots,0} = w_{2k}^c$ the statement is proved.

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